GAUSSIAN PROCESSES
EXERCISE SHEET 12: INFINITE DIMENSIONAL MEASURES II

Exercise 1 (Continuity of Gaussian Processes from Finite Entropy Integral)**.** *Let $G = (G_t)_{t\in T}$ be a centered Gaussian process indexed by a* countable *set T, and let*

$$d(s,t) := \sqrt{\mathbb{E}\big[(G_s - G_t)^2\big]}$$

be its canonical pseudometric. For $\varepsilon > 0$, denote by $N(T, d, \varepsilon)$ the minimal number of d–balls of radius ε needed to cover T. Assume that the Dudley entropy integral is finite:

$$\int_0^\infty \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon < \infty.$$

Show that G is (almost surely) continuous with respect to d.

Exercise 2 (Sandwiching Hilbert spaces and Gaussians on it)**.** *For each real number $k \in \mathbb{R}$, define*

$$A_k := \left\{ \bar{a} = (a_1, a_2, \dots) : \sum_{n=1}^{\infty} a_n^2 n^k < \infty \right\}$$

and equip A_k with the inner product

$$\langle \bar{a}, \bar{b} \rangle_k := \sum_{n=1}^{\infty} a_n b_n n^k.$$

Let $(Z_1, Z_2, \dots)$ be a sequence of i.i.d. standard normal random variables, and for each $a = (a_1, a_2, \dots) \in \ell^2(\mathbb{N})$ define the formal Gaussian

$$G_{\bar{a}} := \sum_{n=1}^{\infty} a_n Z_n.$$

In particular, for every $\bar{a} \in \ell^2(\mathbb{N})$ we may write

$$G_{\bar{a}} = \langle \bar{Z}, \bar{a} \rangle,$$

where $\bar{Z} = (Z_1, Z_2, \dots)$. Note that $||\bar{Z}||^2 = \infty$, so $G_{\bar{a}}$ doesn't naturally make sense.

(a) Show that $(A_k, \langle \cdot, \cdot \rangle_k)$ is a Hilbert space for every $k \in \mathbb{R}$.
(b) Show that for any $\epsilon > 0$, almost surely for all $\bar{a} \in A_{1+\varepsilon}$, we have $G_{\bar{a}} < +\infty$.
(c) Show that for any $\epsilon > 0$, almost surely we have that $\bar{Z} \in A_{-1-\varepsilon}$.